Linear Algebra II

04/07/2012, Wednesday, 14:00-17:00

1 $(10 + 5 = 15)$ **Gram-Schmidt process**

Consider the inner product space $C[-1, 1]$ with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

Let S be the subspace spanned by 1, x, and x^2.

(a) Find an orthonormal basis for S.

(b) Find the best least squares approximation to x^4 by a function from the subspace S.

2 $(3 + 4 + 4 + 4 = 15)$ **Eigenvalues and singular values**

Let A be a real symmetric matrix.

(a) Show that all eigenvalues of A are real.

(b) Show that if λ is an eigenvalue of A then $|\lambda|$ is a singular value.

(c) Show that if x is an eigenvector of A then it is also an eigenvector of $A^T A$.

(d) What can you say about eigenvalues and singular values in case A is positive semi-definite?

3 $(2 + 2 + 3 + 6 + 2 = 15)$ **Diagonalization and singular values**

Let

$$A = \begin{bmatrix} p & q \\ q & p \end{bmatrix}$$

where p and q are real numbers.

(a) Find its eigenvalues and corresponding eigenvectors.

(b) Is it diagonalizable? If yes, diagonalize it.

(c) Find its singular values for $p > q > 0$.

(d) Find its singular value decomposition for $p > q > 0$.

(e) Find the closest (with respect to the Frobenius norm) matrix of rank 1.

4 (15) **Cayley-Hamilton theorem**

Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Find A^{1000} by using the Cayley-Hamilton theorem.

5 $(5+5+5=15)$ **Positive definite matrices**

Let

$$A = \begin{bmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{bmatrix}$$

where a and b are real numbers. Plot the regions of (a,b)-plane in which it is

(a) positive definite.

(b) negative definite.

(c) indefinite.

6 $(2+2+3+5+3=15)$ **Jordan canonical form**

(a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

(b) How many linearly independent eigenvectors does the matrix A have?

(c) Is it diagonalizable? Why?

(d) Put it into the Jordan canonical form.

(e) Compute

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} A^k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

---

10 pts gratis.